Power of 2 in the Cantor Set

Chris Cai/Linrong Cai
University of Wisconsin Madison

Jordan Ellenberg
University of Wisconsin Madison

University of Wisconsin Madison

Date: December 11, 2022

Abstract

This paper takes a number theory view on an optional homework problem [2] given by Professor Ellenberg. The probelm is for which positive integers k is $\frac{1}{2^k}$ contained in the Cantor set. In the original problem, we showed that $\frac{1}{2}$ and $\frac{1}{8}$ are not in the Cantor set, while $\frac{1}{4}$ is in the Cantor set. This paper will show that in the general case for $\frac{1}{2^k}$, only $\frac{1}{4}$ is in the Cantor set.

Keywords: Cantor Set, Power of 2, Number Theory, Analysis

1 Introduction

1.1 Cantor Set

The Cantor set [1] is $\mathcal{C}$ is created by iteratively deleting the open middle third from a set of line segments, starting from the interval [0,1]. The Cantor set contains all points in the interval [0, 1] that are not deleted at any step in this infinite process. This process can be described recursively by setting

$$C_0 := [0, 1]$$

and

$$C_n = \frac{1}{3}(C_{n-1} \cup (2 + C_{n-1}))$$

$$\mathcal{C} = \lim_{n\to\infty} C_n = \bigcap_{n=0}^{\infty} C_n$$

The explicit closed formulas for the Cantor set are

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{a=0}^{3^n-1} (\frac{3a+1}{3^{n+1}}, \frac{3a+2}{3^{n+1}}) \tag{1}$$

1.2 $\frac{1}{4}$ is in the Cantor set

This section will show that $\frac{1}{4}$ is in the Cantor set with basic number theory knowledge. Suppose it is not in the Cantor set towards a contradiction. Then by (1), exist a and n such that

$$\frac{3a+1}{3^{n+1}} < \frac{1}{4} < \frac{3a+2}{3^{n+1}}$$
$$12a+4 < 3^{n+1} < 12a+8$$

The only possible value for 3^{n+1} to take are $12a+5, 12a+6, 12a+7$. Among them $12a+6$ is the only term that is divisible by 3. As a result, if such a, n exist then $3^{n+1} = 12a+6$. However, 3^{n+1} is odd and $12a+6$ is even, which is a contradiction. This showed that $\frac{1}{4}$ has to be in the Cantor Set.

2 Preliminiary

This section present several lemma that is needed for the later proof.

Lemma 2.1. *The smallest positive integer p such that* $3^p \equiv 1 \pmod{2^k}$ *is* 2^{k-2} *for* $k > 2$. *The multiplicative order of* 3 *modulo* 2^k *is* 2^{k-2} *for* $k > 2$.

Proof. From Euler's Theorem, since 3 and 2^k are coprime, $3^{\varphi(2^k)} \equiv 1 \pmod{2^k}$. $\varphi(2^k) = 2^{k-1}$, so we have $3^{2^{k-1}} \equiv 1 \pmod{2^k}$. However, 2^{k-1} is not the smallest positive integer p such that $3^p \equiv 1 \pmod{2^k}$. In fact, 2^{k-2} is the order of 3 modulo 2^k. In order to prove this, it suffices to show that $3^{2^{k-2}} \equiv 1 \pmod{2^k}$ and $3^{2^{k-3}} \not\equiv 1 \pmod{2^k}$. Use induction to show $3^{2^{k-2}} \equiv 1 \pmod{2^k}$ for $k > 2$, base case $3^2 \equiv 1 \pmod{2^3}$. Induction step: suppose $3^{2^{k-2}} \equiv 1 \pmod{2^k}$. Since $3^{2^{(k+1)-2}} - 1 = (3^{2^{k-2}})^2 - 1 = (3^{2^{k-2}}+1) \times (3^{2^{k-2}}-1)$. Because $2^k \mid 3^{2^{k-2}} - 1$ and $2 \mid 3^{2^{k-2}} + 1 \Rightarrow 2^{k+1} \mid 3^{2^{(k+1)-2}} - 1$. As a result, the induction hypothesis holds. Therefore we have

$$3^{2^{k-2}} \equiv 1 \pmod{2^k} \quad k > 2 \tag{2}$$

Similarly, we can use induction to prove $3^{2^{k-3}} \not\equiv 1 \pmod{2^k}$. Base case: $3^1 \not\equiv 1 \pmod{2^3}$. Induction step: $3^{2^{(k+1)-3}} - 1 = (3^{2^{k-3}}+1) \times (3^{2^{k-3}}-1)$. From (2) we know that $3^{2^{k-3}} = 2^{k-1}a+1$, for some positive integer a. Therefore $3^{2^{k-3}} + 1 = 2^{k-1}a + 2$, which is divisible be 2 but not 4. $2^k \nmid 3^{2^{k-3}} - 1 \Rightarrow 2^{k+1} \nmid 3^{2^{k+1-3}} - 1$. As a result, the induction hypothesis holds. Therefore we have

$$3^{2^{k-3}} \not\equiv 1 \pmod{2^k} \quad k > 2 \tag{3}$$

(2) and (3) force the multiplicative order of 3 modulo 2^k is 2^{k-2} for $k > 2$. □

Corollary 2.2. *The above lemma implies that* 3^n *has* 2^{k-2} *distinct remainders modulo* 2^k *for* $k > 2$.

Lemma 2.3. $(2^k - 1)3^n$ *has* 2^{k-2} *distinct remainders and none of them is the same with the remainders of* 3^n *for* $k > 2$.

Proof. $(2^k-1)3^n$ has 2^{k-2} distinct remainders is because (2^k-1) is provides an injective mapping from the remainders of 3^n to remainders of $(2^k-1)3^n$ and **2.2**. Suppose $\exists n$ such that $(2^k-1)3^n$ has the same remainders with 3^n towards a contradiction. Then $2^k \mid 3^n-(2^k-1)3^n$, but $3^n-(2^k-1)3^n \equiv 2\cdot 3^n \not\equiv 0 \pmod{2^k}$ for k >2, which gives a contradiction. As a result the lemma holds. □

Corollary 2.4. *3^n and $(2^k-1)3^n$ together has 2^{k-1} distinct remainders modulo 2^k for $k>2$.*

3 Kyle Conjecture

Theorem 3.1. *None of the powers of 2 for $n<-2$, are in the Cantor set. [3]*

This conjecture is provided by Kyle Horton on Piazza, classmates from Math521. Here is a Python program that I wrote to help find which interval in the complement of Cantor set does $\frac{1}{2^k}$ belongs to, which further supports this conjecture.

```
for p in range (1,16):
    for k in range (100000):
        found = False
        a = math.pow(2,p)*3*k+math.pow(2,p)
        b = math.pow(2,p)*3*k+math.pow(2,p+1)
        for j in range (int(a+1),int(b)):
            if math.log10(j)/ math.log10(3)%1 == 0:
                print(f'For 1/2^{p}: {3*k+1}/3^{int(math.log10(j)/ math.log10(3)
                    )} < 1/{int(math.pow(2,p))} < {3*k+2}/3^{int(math.log10(j)/
                    math.log10(3))}')
                found = True
                break
        if found:
            break
For 1/2^1: 1/3^1 < 1/2 < 2/3^1
For 1/2^3: 1/3^2 < 1/8 < 2/3^2
For 1/2^4: 1/3^3 < 1/16 < 2/3^3
For 1/2^5: 7/3^5 < 1/32 < 8/3^5
For 1/2^6: 1/3^4 < 1/64 < 2/3^4
For 1/2^7: 1/3^5 < 1/128 < 2/3^5
For 1/2^8: 25/3^8 < 1/256 < 26/3^8
For 1/2^9: 1/3^6 < 1/512 < 2/3^6
For 1/2^10: 19/3^9 < 1/1024 < 20/3^9
For 1/2^11: 1/3^7 < 1/2048 < 2/3^7
For 1/2^12: 1/3^8 < 1/4096 < 2/3^8
For 1/2^13: 7/3^10 < 1/8192 < 8/3^10
For 1/2^14: 1/3^9 < 1/16384 < 2/3^9
For 1/2^15: 1/3^10 < 1/32768 < 2/3^10
```

4 Proof for Kyle Conjecture

Proof. From (1), we know that suppose $\frac{1}{2^k}$ is not in the Cantor set. Then exist n and a such that

$$\frac{3a+1}{3^{n+1}} < \frac{1}{2^k} < \frac{3a+2}{3^{n+1}}$$

$$2^k a + \frac{2^k}{3} < 3^n < 2^k a + \frac{2^{k+1}}{3} \tag{4}$$

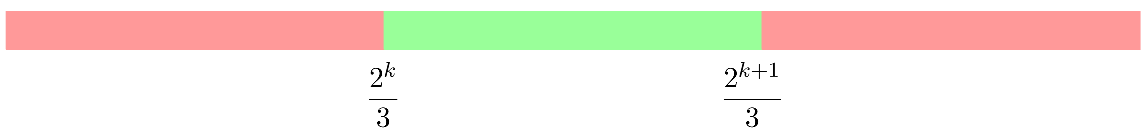
Let r_1 be the remainder of 3^n divided by 2^k, we take the remainder of (4) divided by 2^k and get:

$$\frac{2^k}{3} < r_1 < \frac{2^{k+1}}{3} \tag{5}$$

Cantor set is symmetric about $\frac{1}{2}$, so if $\frac{1}{2^k}$ is not in the Cantor set, then $\frac{2^k-1}{2^k}$ is also not in the Cantor set. As a result, let r_2 be the remainder of $(2^k-1)3^n$ divided by 2^k we get:

$$\frac{2^k}{3} < r_2 < \frac{2^{k+1}}{3} \tag{6}$$

Let's draw the remainder when divided by 2^k in the following picture.



□

Suppose the number of integers in the red region that r_1 and r_2 can take is less than the possible remainders r_1 and r_2 can take. Then that means, there has to be remainders of r_1 and r_2 falling in to the green region, which implies that exist n such that the remainder of 3^n or $(2^k-1)3^n$ is between $\frac{2^k}{3}$ and $\frac{2^{k+1}}{3}$. In addition, since a can take value up to $3^n - 1$, there has to be a such that (4) holds.

From corollary 2.4 that is derived from lemma 2.1 and lemma 2.3, we know that r_1 and r_2 can take 2^{k-1} distinct integer value. There are $\lceil \frac{2^k}{3} \rceil$ integer in the first red region including 0, $\lfloor \frac{2^k}{3} \rfloor$ integer in the second red region not including 2^k. Together this gives $2 \cdot \lfloor \frac{2^k}{3} \rfloor + 1$ integers. However since r_1 and r_2 can only be odd. Due to the fact that both 3^n and $(2^k-1)3^n$ are odd, the number of possible value in the red region that r_1 and r_2 can take are $\frac{2 \cdot \lfloor \frac{2^k}{3} \rfloor + 1 + (-1)^k}{2}$ numbers, which is smaller than $\frac{2^k}{3} + 1$. Since $2^{k-1} > \frac{2^k}{3} + 1$ for all $k > 2$. There exists n, such that r_1 or r_2 are forced to be in the green region, and from the above explaination, this showed that exist n and a such that:

$$\frac{3a+1}{3^{n+1}} < \frac{1}{2^k} < \frac{3a+2}{3^{n+1}} \tag{7}$$

This showed that $\frac{1}{2^k}$ are not in the Cantor set for $k > 2$, together with **1.2** we showed that the only power of 2 in the Cantor set is $\frac{1}{4}$.

5 Discussion

Theorem 3.1 showed that $\frac{1}{2^{q+1}}$ are always not in the Cantor set for $q > 1$ suggests that there always exist c and p such that :

$$(3c+1)2^{q+1} < 3^{p+1} < (3c+2)2^{q+1} \tag{8}$$

Which is equivalent to;

$$|3^p - (2c+1)2^q| < 2^q \tag{9}$$

This somehow suggests the powers of 3 and the powers of 2 should not be too separated. This remind me of the separation between powers of 2 and powers of 3 which is the following.

Proposition 5.1. *Separation between powers of 2 and powers of 3*

For any postive integers p, q one has

$$|3^p - 2^q| \geq \frac{c}{q^C} 3^p$$

[4]

(9) and Proposition 5.1 is sort of opposite of each other, and they together force the powers of 3 and the powers of 2 to be in a intertwined relation with each other.

6 Acknowledgement

Special thanks to an Unkown Question Answerer from a screenshot. The two lemma mentioned in this paper is suggested in the screenshot, I help to fill in the proof of this two lemma. The later part of the screenshot proof have some mistakes but are fixed in this paper.

I found Proposition 5.1 from Terence Tao's Blog when searching distance between powers of 2 and powers of 3. Interestingly, I also found Professor Ellenberg's Blog Quomodocumque pined on the left topic bar of Tao's blog. Then I saw the blog Professor Ellenberg posted about finishing teaching Analysis, which also leads to the final of this paper. It's so nice to have you as our analysis professor, this is certainly an unforgettable experience, and I certainly shouldn't forget because the knowledge is so useful. Thank you Professor Ellenberg!

Bibliography

[1] Wikipedia contributors. "Cantor set". In: (2022).

[2] Jordan Ellenberg. "Homework 8". In: *2022 Fall Math 521 Homework 8* (2022).

[3] Kyle Horton. "Piazza Discussion". In: *2022 Fall Math 521 Homework 8 Piazza Discussion* (2022).

[4] Terence Tao. "Hilbert's seventh problem, and powers of 2 and 3". In: *Tao's Blog* (2011).